Homework 6

Section 1.3

12) We have \( \pi_1(S' \times S') = \langle a, b \rangle \), call this \( G \). Let \( H \) be the normal subgroup generated by \( a^2, b^2, \) and \( (ab)^4 \). That is, 

\[ H = \langle a^2, b^2, (ab)^4 \rangle. \]

Consider \( X_H \) below:

![Diagram of \( X_H \)]

Let \( p : X_H \to S' \times S' \) be the obvious covering map. By Corollary 2 (iii) we know the image subgroup consists of the homotopy classes of the loops in \( X \) whose lifts to \( X_H \) are loops. Let \( Y \) be a non-trivial loop in \( p_\ast(\pi_1(X_H)) \). Then \( Y \) necessarily comprises a "walk" around \( X_H \), some combination of loops \( a^2, b^2 \), or \( (ab)^4 \), and a "walk back" to its basepoint. But this is the conjugate of some element in \( \langle a^2, b^2, (ab)^4 \rangle \) so \( p_\ast \leq H \). Let \( h \in H \). Then \( h = \omega x \omega^{-1} \) for some \( \omega \in G \), \( x \in \langle a^2, b^2, (ab)^4 \rangle \). We can see that \( \omega \) will travel in some way around \( X_H \) (via \( \omega \)), then form a loop at some vertex (via \( x \)), then retrace its steps back to the basepoint (via \( \omega^{-1} \)). This is a loop in \( X_H \) so \( \tilde{p}_\ast(\pi_1(X_H)) \). Hence \( p_\ast(\pi_1(X_H)) = H \leq G \).

(Note that this works because \( X_H \) is symmetric about its vertices.)

Since \( S' \times S' \) is path-connected and locally simply-connected we know by the theorem from 2.12 that \( X_H \) is unique up to isomorphism. Therefore, this \( X_H \) is correct.
We will continue by considering the universal abelian covers of $X$.

Let $H$ be a subgroup such that $YH$ is abelian. Then $H$ is necessarily normal in $G$ and $G/H$ is essential abelian. But the group $G/H$ is an abelian cover of $YH$ by the unique statement of the theorem. So $A = G/H$.

The universal abelian cover $\pi : \overline{A} \to X$ is path-connected. Observe for any $[\overline{A}] = \overline{A}$ and $F = \{ \overline{a} \} \subseteq \overline{A}$.

Thus, for any $[\overline{a}] = \overline{a}$ and $F = \{ \overline{a} \} \subseteq \overline{A}$, the group $\pi G/F$ is abelian.

So we indeed have an abelian cover. Note that we get the uniqueness of $\overline{A}$ by the unique statement of the theorem. From this $\overline{A}$.
In the case of $S' \times S'$, $\pi_1(S' \times S') = \langle a, b \rangle$. Then the commutator subgroup is $\langle [a, b] : g$ and $h$ are words in $a$ and $b \rangle$. The covering space corresponding to this is

Any commutator will traverse a loop in this covering space. For example $[ab^2, babab] = (ababab)(a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1})$ is depicted.

In the case of $S' \times S' \times S'$, the universal abelian cover will be the "integer grid" in $\mathbb{R}^2$. 
We will construct nonnormal covering spaces of the Klein bottle $K^2$ by a Klein bottle and by a torus.

The cell complex of $K^2$ is $\phi \in \mathbb{R}^2$.

Thus $\pi_1(K^2) = \langle a, b \mid abab^{-1} = 1 \rangle$.

We know from class (4/6) that the universal cover of $K^2$ is $\mathbb{R}^2$ with the universal covering action depicted in the identification below:

Consider the subgroup of $\pi_1(K^2)$ generated by $a^3$ and $b$; that is, $\langle a^3, b \rangle$. Note that this subgroup is not normal because $ba^3 = a(ab)b^{-1}$ and $aba^{-1} = a(ba^{-1})$.

We used $ba = ab$ which follows from the relation $abab^{-1} = 1$.

The covering space corresponding to $\langle a^3, b \rangle$ is below:

After some identifying, we have

Which is a Klein bottle.

Consider the subgroup $\langle a^3, ab^2 \rangle \subset \pi_1(K^2)$. From the relation $abab^{-1} = 1$, we know $ab = ba$ and $b = aba$. The product of these yields $ab^2 = ba'aba = b^2a$, so $a$ and $b^2$ commute. Hence, $a^3$ and $ab^2$ commute, so the covering space corresponding to $\langle a^3, ab^2 \rangle$ is a torus (since $\pi_1(K^2) = \mathbb{Z} \times \mathbb{Z}$). We must check that $\langle a^3, ab^2 \rangle$ is not normal. This follows from $a^3 \notin \langle a^3, ab^2 \rangle$ and $ba^3 = ba(ab^{-1}) = ba^2(ba) = ba(aba) = ba(ab)a = ba \notin \langle a^3, ab^2 \rangle$.

Thus, the covering space of $K^2$ corresponding to $\langle a^3, ab^2 \rangle$ is not normal.

This tells us that the covering space that $\pi_1$ maps to $\pi_1(K^2)$ is why does it have the
Let $X$ be the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{0\}$ in the torus. We will compute $\pi_1(X)$, describe the universal cover of $X$, and describe the action of $\pi_1(X)$ on the universal cover.

\[ \pi_1(T^2) = \langle a, b | ab = ba \rangle \]

\[ \pi_1(M) \cong \mathbb{Z} \]

We can apply the van Kampen theorem to get

\[ \pi_1(X) = \pi_1(S^1 \times S^1) \ast \pi_1(M) \]
\[ = \langle a, b, c | ab = ba, a = c^2 \rangle \]
\[ = \langle b, c | c^2, b = be^2 \rangle \].

The universal cover of $M$ is $\mathbb{R}^2$ with strips homeomorphic to the Möbius band attached at the endpoints of the lifts of $b$. On top of each strip is attached a half-plane orthogonal to the strip. These planes have strips attached, etc. The planes cover the torus in the usual way and the strips cover the Möbius Band. It is simply-connected because it is clearly path-connected and any loop can homotope through the planes and through the strips (which are each simply-connected) to the constant loops.

Action: the element $b$ translates a given point one unit in the positive direction orthogonal to the strips. If the point is on a strip, $b$ shifts it to the corresponding point on the next strip. The element $c$ translates one half unit in the positive direction that the strips lie, and moves the plane to the next plane attached to the strip.
Let $Y$ be the space obtained by attaching a Möbius band to $\mathbb{R}P^2$ via a homeomorphism from its boundary circle to a circle in $\mathbb{R}P^2$ lifting to the equator in a covering space $S^2$ of $\mathbb{R}P^2$.

\[
\begin{align*}
\pi_1(\mathbb{R}P^2) &= \langle a, b^2 = 1 \rangle \\
\pi_1(M) &= \langle c \rangle
\end{align*}
\]

By van Kampen,

\[
\pi_1(Y) = \pi_1(\mathbb{R}P^2) \ast \pi_1(M) = \langle a, c \mid a^2 = 1, a = c^2 \rangle
\]

\[
\cong \mathbb{Z}/4\mathbb{Z}.
\]

I believe the universal cover of $Y$ would be a pair of spheres, one inside the other, with a strip homeomorphic to the Möbius band similar to a bridge from the smaller sphere to the larger sphere (like a belt around the inside sphere). Unfortunately, I am unable to intuit the covering action of $\pi_1(Y)$ on the universal cover. Though my best guess is that $c$ acts by quarter rotation.

This obviously has 4-sheets for $\mathbb{R}P^2$ with the 4 hemispheres and the strip attached at the equator would have to 4-fold cover the Möbius band.