Sharp Sobolev–Strichartz estimates for the free Schrödinger propagator

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Abstract. We consider gaussian extremisability of sharp linear Sobolev–Strichartz estimates and closely related sharp bilinear Ozawa–Tsutsumi estimates for the free Schrödinger equation.

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1. Introduction

For $d \geq 1$ and $s \in [0, \frac{d}{2})$, it is well-known that the solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ of the free Schrödinger equation
\[ i\partial_t u + \Delta u = 0, \quad u(0) = u_0 \in \dot{H}^s(\mathbb{R}^d) \] (1.1)
satisfies the global space-time estimate
\[ \|u\|_{L^p(d, s)} \leq S(d, s)\|u_0\|_{\dot{H}^s} \] (1.2)
for some finite constant $S(d, s)$, which we assume to be the optimal (i.e. smallest) such constant. Here, $p(d, s) = \frac{2(d+2)}{d-2s}$ and, as usual, $\dot{H}^s(\mathbb{R}^d)$ denotes the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2}f\|_{L^2}$. This article will be concerned with optimal constants and extremisers, and we note immediately that $S(d, s)$ and the shape of corresponding extremisers are only known in the rather special cases $(d, s) \in \{(1, 0), (2, 0)\}$ (see Foschi [5] and also Hundertmark–Zharnitsky [6]). In such cases, the isotropic gaussian initial data $u_0(x) = \exp(-|x|^2)$ is an extremiser.

Very closely related are the sharp bilinear estimates
\[ \|(-\Delta)^{\frac{d-4}{4}}(uv)\|_{L^2} \leq OT(d)\|u_0\|_{L^2}\|v_0\|_{L^2} \] (1.3)
due to Ozawa–Tsutsumi [7], where $d \geq 2$ and
\[ OT(d) = \Gamma(\frac{d}{2})^{-\frac{1}{2}}2^{-\frac{d}{4}}\pi^{\frac{d-4}{4}}. \]
In (1.3), $u$ and $v$ are solutions of (1.1) with square-integrable initial data $u_0$ and $v_0$. The constant $\mathbf{O}(d)$ is optimal and $(u_0, v_0)$ is an extremising pair when $u_0(x) = v_0(x) = \exp(-|x|^2)$.

The sharp estimate (1.3) was motivated by the case of one spatial dimension, in which case (1.3) is an identity

$$\|(-\Delta)^{\frac{1}{4}}(uv)\|_{L^2} = \mathbf{O}(1)\|u_0\|_{L^2}\|v_0\|_{L^2}. \tag{1.4}$$

This basic tool was established and used in [7] to prove local-wellposedness for certain nonlinear Schrödinger equations with nonlinearities including $\partial(|u|^2)u$ and initial data in $H^{1/2}$. Thus, (1.4) gives control on the null gauge form $\partial(uv)$ for the one-dimensional Schrödinger equation, and (1.3) gives estimates for the null gauge form in higher dimensions.

We remark that taking $d = 2$ and $u_0 = v_0$ in (1.3) immediately yields the optimal constant $\mathbf{S}(2, 0)$ and its gaussian extremisability (this was not explicitly observed in [7]). The approach in [7] is different to the approaches in [5] and [6], so this provides an alternative derivation of this optimal constant.

As far as we know, for the Sobolev–Strichartz estimate (1.1), no conjecture has been made on the shape of extremising initial data in the case where $s$ is strictly positive. Extremising initial data certainly exist for all admissible $d \geq 1$ and $s \in [0, \frac{d}{2})$ (see, for example, [8]). In this direction, our first observation is the following.

**Theorem 1.1.** Only if $s = 0$ are gaussians $u_0$ such that

$$\hat{u}_0(\xi) = \exp(a|\xi|^2 + ib \cdot \xi + c)$$

for some $a, c \in \mathbb{C}, b \in \mathbb{R}^d, \text{Re}(a) < 0$, critical points for the functional

$$u_0 \mapsto \frac{\|e^{it\Delta}u_0\|_{L^p(d,s)}}{\|u_0\|_{\dot{H}^s}}. \tag{1.5}$$

Theorem 1.1 of course implies that gaussians are not amongst the class of extremisers for (1.2) for any admissible $s$ which are strictly positive; i.e. $s \in (0, \frac{d}{2})$. We find this outcome particularly interesting when measured against the analogous sharp estimates for the wave propagator $e^{it\sqrt{-\Delta}}$. Here, it is known that if $d \geq 2$ and $s \in [0, \frac{d-1}{2})$ then

$$\|u\|_{L^p(d-1,s)} \leq \mathbf{W}(d, s)\|u_0\|_{\dot{H}^{s+\frac{1}{2}}} \tag{1.6}$$

for all solutions of the (pseudo) wave equation

$$i\partial_t u + \sqrt{-\Delta}u = 0, \quad u(0) = u_0 \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d).$$

Again, we take $\mathbf{W}(d, s)$ to be the optimal constant, which is finite for the given range of parameters $(d, s)$. It is known that initial data $u_0$ for which

$$\hat{u}_0(\xi) = |\xi|^{-1} \exp(-|\xi|^2) \tag{1.7}$$

are extremisers for (1.6) when $(d, s) \in \{(2, 0), (3, 0), (5, \frac{1}{2})\}$ (uniquely, up to certain transformations). The cases $(2, 0)$ and $(3, 0)$ were established by Foschi [5] and the case $(5, \frac{1}{2})$ was established in [2]. Thus, $u_0$ satisfying (1.7)
is an extremiser for $W(d,s)$ for two distinct values of $s$. Theorem 1.1 shows that this phenomena does not occur for $S(d,s)$ and gaussian $u_0$.

The condition $s = 0$ is, in fact, necessary and sufficient for gaussians to be critical points for the functional in (1.5). The sufficiency part was demonstrated by Hundertmark–Zharnitsky [6]. See also the work of Christ–Quilodrán [3] where a closely related result to Theorem 1.1 was established in the context of adjoint Fourier restriction estimates for the paraboloid; in fact, we prove Theorem 1.1 by making small modifications to their argument.

For the cases $(d,s) \in \{(1,0),(2,0)\}$, it is known that the isotropic gaussian $u_0(x) = \exp(-|x|^2)$ is, up to certain transformations, the only extremising initial data for (1.1); see [5], [6]. Furthermore, it is conjectured ([5], [6]) that gaussians are the only extremisers for $S(d,0)$ for all $d \geq 1$. Providing a full characterisation of the set of extremisers often requires delicate arguments, and applications of sharp estimates frequently demand that such a characterisation is established. For example, recent work of Duyckaerts–Merle–Roudenko [4] considered extremisers for the global Strichartz norm

$$i\partial_t u + \Delta u + \gamma |u|^4 u = 0,$$  

where $\gamma = 1$ in the focusing case and $\gamma = -1$ in the defocusing case. In particular, it was shown in [4] that for $\delta > 0$ sufficiently small,

$$I(\delta) := \sup_{\|u\|_2 = \delta} \|u\|_{L^p(d,0)}$$

is attained for some initial data $u_0(\delta) \in L^2(\mathbb{R}^d)$ with $\|u_0(\delta)\|_2 = \delta$. When $d = 1,2$ they prove significantly more; it is shown that, as $\delta \to 0$,

$$I(\delta) = S(d,0)\delta + \gamma \Lambda(d)\delta^{1+\frac{4}{d}} + O(\delta^{1+\frac{5}{d}}),$$

where $\Lambda(d)$ is some positive constant, and that any extremising initial data $u_0(\delta)$ is, for $\delta$ sufficiently small and up to certain transformations, close to $\delta G_0$, where $G_0$ is the isotropic centred gaussian which has been $L^2$-normalised. For this additional information concerning $I(\delta)$ when $d = 1,2$, it was vital to know a full characterisation of the extremisers for $S(d,0)$.

Our next result establishes a full characterisation of extremisers for the bilinear Sobolev–Strichartz estimate (1.3) of Ozawa–Tsutsumi; this question was left open in [7] and the following theorem says that extremisers for (1.3) must be isotropic centred gaussians, up to certain transformations.

**Theorem 1.2.** Let $d \geq 2$. We have equality in the estimate (1.3) if and only if there exist $a,c,d \in \mathbb{C}$, $b \in \mathbb{C}^d$, with $\text{Re}(a) < 0$, so that

$$u_0(x) = \exp(a|x|^2 + b \cdot x + c), \quad v_0(x) = \exp(a|x|^2 + b \cdot x + d). \quad (1.8)$$

**2. Further remarks and proofs**

For $d \geq 1$, $q \in (1,\frac{2(d+1)}{d})$ and $p = \frac{(d+2)q'}{d}$, it was shown in [3] that gaussians are critical points for the $L^q(\mathbb{R}^d, d\sigma) \to L^p(\mathbb{R}^{d+1})$ adjoint Fourier restriction
estimates associated to the paraboloid
\[ \mathbb{P}^d = \{(\xi, |\xi|^2) : \xi \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1} \]
if and only if \( q = 2 \). Here, \( d\sigma \) is the measure supported on \( \mathbb{P}^d \) given by
\[ \int_{\mathbb{P}^d} F \, d\sigma := \int_{\mathbb{R}^d} F(\xi, |\xi|^2) \, d\xi \]
and \( d\xi \) is Lebesgue measure on \( \mathbb{R}^d \). A mixed-norm generalisation of this is also established in [3] and we remark that Theorem 1.1 may also be extended by measuring the solution in appropriate \( L^p_t L^q_x(\mathbb{R}^{d+1}) \) norms.

To prove Theorem 1.1 we make minor modifications to the argument in [3] associated with replacing \( L^q(\mathbb{P}^d, d\sigma) \) by \( H^s(\mathbb{R}^d) \).

**Proof of Theorem 1.1.** We fix \( d \geq 1, s \in (0, \frac{d}{2}) \) and let \( p = p(d, s) \). If \( \Psi \) is the functional
\[ \Psi(u_0) = \frac{\|e^{it\Delta}u_0\|_{L^p}^p}{\|u_0\|_{H^s}^{p}}, \]
defined for nonzero \( u_0 \in \dot{H}^s(\mathbb{R}^d) \), then \( u_0 \) is a critical point if
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\Psi(u_0 + \varepsilon v_0) - \Psi(u_0)) = 0 \]
for any \( v_0 \in \dot{H}^s(\mathbb{R}^d) \), where \( \varepsilon \) is a complex parameter. For brevity, we write \( u(t, \cdot) = e^{it\Delta}u_0 \) and \( v(t, \cdot) = e^{it\Delta}v_0 \).

Using Lemma 2.3 of [3], which gives an expansion of \( \|F + \varepsilon G\|_{L^p}^p \) as \( \varepsilon \to 0, \varepsilon \in \mathbb{C} \), we obtain some constant \( \gamma > 1 \) such that
\[ \|u + \varepsilon v\|_{L^p}^p = \|u\|_{L^p}^p + p \int_{\mathbb{R}^{d+1}} |u(t, x)|^p \text{Re}(\varepsilon \frac{v(t, x)}{u(t, x)}) \, dx \, dt + O(|\varepsilon|^\gamma) \]
and
\[ \|u_0 + \varepsilon v_0\|_{H^s}^{p} = \|u_0\|_{H^s}^{p} + (2\pi)^d p \|u_0\|_{H^s}^{p-2} \text{Re} \left( \varepsilon \int_{\mathbb{R}^d} \hat{u}_0(\xi) \bar{\hat{v}_0}(\xi) |\xi|^{2s} \, d\xi \right) + O(|\varepsilon|^\gamma) \]
as \( \varepsilon \to 0 \). It then follows that \( u_0 \) is a critical point if and only if there exists \( \lambda > 0 \) such that
\[ \int_{\mathbb{R}^{d+1}} |u(t, x)|^{p-2} u(t, x) \exp(-i(x \cdot \xi - t|\xi|^2)) \, dx \, dt = \lambda |\xi|^{2s} \hat{u}_0(\xi) \tag{2.1} \]
for almost all \( \xi \in \mathbb{R}^d \). For Theorem 1.1, it suffices to show that \( u_0 \) is not a critical point, where \( u_0(x) = \exp(-\frac{1}{4}|x|^2) \). This reduction follows because \( u_0 \) such that \( \hat{u}_0(\xi) = \exp(a|\xi|^2 + ib \cdot \xi + c) \), with \( a, c \in \mathbb{C} \) and \( b \in \mathbb{R}^d \), can be generated from a centred isotropic gaussian under the action of the group generated by:

1. **space-time translations**: \( u(t, x) \to u(t + t_0, x + x_0) \) with \( (t_0, x_0) \in \mathbb{R}^{d+1} \);
2. **parabolic dilations**: \( u(t, x) \to u(\mu^2 t, \mu x) \) with \( \mu > 0 \);
3. **change of scale**: \( u(t, x) \to \mu u(t, x) \) with \( \mu > 0 \);
4. **phase shift**: \( u(t, x) \to e^{i\theta} u(t, x) \) with \( \theta \in \mathbb{R} \).
The Euler–Lagrange equation (2.1) is invariant under each of the above actions.

For \( u_0(x) = \exp(-\frac{1}{4}|x|^2) \) we have \( \hat{u}_0(\xi) = C_d \exp(-|\xi|^2) \) and

\[
u(t, x) = C_d \frac{1}{(1 + it)^\frac{d}{2}} \exp \left( -\frac{|x|^2}{4(1 + it)} \right)
\]

for some positive constants \( C_d \) (which may differ). Thus, (2.1) is equivalent to

\[
\text{I}(a) = C_{d,s} a^s
\]

for all \( a \in [0, \infty) \), where \( C_{d,s} \) is some positive constant,

\[
\text{I}(a) := \int_{\mathbb{R}} \frac{H(t)}{(1 + it)^\frac{d}{2}(p-2)} \exp \left( a \frac{(p-2)(1 + it)}{p - 1 - it} \right) dt
\]

and

\[
H(t) := (1 - it)^{-\frac{d}{4}(p-4)}(p - 1 - it)^{-\frac{d}{4}}.
\]

A power series expansion of the exponential term leads to

\[
\text{I}(a) = \sum_{j=0}^{\infty} \frac{(p-2)^j I_j}{j!} a^j,
\]

where

\[
I_j := \int_{\mathbb{R}} (1 + it)^j \frac{-d(j+1)}{d-2s} H_j(t) dt
\]

and

\[
H_j(t) = (1 - it)^{-\frac{d}{4}(p-4)}(p - 1 - it)^{-\frac{d}{4} - j}.
\]

Since \( H_j \) is holomorphic in the upper half-plane and

\[
|(1 + it)^j \frac{-d(j+1)}{d-2s} H_j(t)| \leq C |t|^{-\frac{2d(j+1)}{d-2s}},
\]

with \( \frac{2d(j+1)}{d-2s} > 1 \), it follows (using Lemma 4.1 of [3]) that for \( j > \frac{d(j+1)}{d-2s} - 1 \) we have

\[
I_j = -2 \sin(\gamma_j \pi) \int_{0}^{\infty} r^{\gamma_j} H_j(i + ir) dr,
\]

where \( \gamma_j := j - \frac{d(j+1)}{d-2s} \). Since \( H_j(i + ir) > 0 \) for all \( r \geq 0 \), it is clear that \( I_j = 0 \) if and only if \( \gamma_j \in \mathbb{Z} \).

In the case where \( s \in (0, \frac{d}{2}) \cap \mathbb{N} \), using (2.2), (2.3) and a power series uniqueness argument, it follows that \( I_j = 0 \) for all \( j \neq s \) and \( I_j \neq 0 \) for \( j = s \). If, additionally, \( \frac{d(j+1)}{d-2s} \notin \mathbb{N} \), then (2.4) implies \( I_j \neq 0 \) for any \( j > \max\{\frac{d(j+1)}{d-2s} - 1, s\} \), which gives a contradiction. If, instead, \( \frac{d(j+1)}{d-2s} \in \mathbb{N} \), then for \( j_* = \frac{d(j+1)}{d-2s} - 1 \) we may use Cauchy’s residue theorem to obtain

\[
I_{j_*} = \int_{\mathbb{R}} (1 + it)^{-1} H_{j_*}(t) dt = 2\pi H_{j_*}(i) \neq 0.
\]

Since \( s > 0 \) we have \( j_* \neq s \) and so this is also a contradiction.
In the remaining case where \( s \in (0, \frac{d}{2}) \) and \( s \notin \mathbb{N} \), one can see that (2.2) cannot hold for all \( a \in [0, \infty) \) since (2.3) implies that \( I(a) \) is \( k \) times (right) differentiable at \( a = 0 \) for each \( k \in \mathbb{N} \), whereas \( a \mapsto a^s \) is not. \( \square \)

Regarding Theorem 1.2, we begin with the observation that the proof of (1.3) in [7], involving several well-chosen changes of variables, leads to the representation

\[
\|(-\Delta)^{\frac{d}{2}}(u\tilde{v})\|_{L^2}^2 = C_d \int_{\mathfrak{M}} \left| \int_{\mathbb{R}^d} \hat{u}_0((r-p)\omega-\eta)\hat{v}_0(\eta) \, d\Sigma_{\omega,r}(\eta) \right|^2 \, d\sigma(\omega) \, dr \, dp,
\]

where \( \mathfrak{M} = S^{d-1} \times \mathbb{R}^2 \), \( d\Sigma_{\omega,r}(\eta) = \delta(r - \omega \cdot \eta) \, d\eta \), \( \delta \) is the Dirac measure on \( \mathbb{R} \) supported at the origin, and \( d\sigma \) is the induced Lebesgue measure on \( S^{d-1} \). The constant \( C_d \) is explicitly computable (and whose value depends on the chosen convention for the Fourier transform).

An application of Cauchy–Schwarz with respect to the measure \( d\Sigma_{r,\omega} \) for each fixed \( (\omega,r,p) \in \mathfrak{M} \) yields (1.3). Using the standard fact that equality holds in the Cauchy–Schwarz inequality precisely when the constituent functions are linearly dependent, we see that if \( (u_0, v_0) \) is an extremising pair, then there exists a scalar function \( \Lambda \) such that

\[
\hat{u}_0((r-p)\omega-\eta) = \Lambda(\omega,r,p) \overline{\hat{v}_0(\eta)} \tag{2.5}
\]

for almost all \( \eta \in \mathbb{R}^d \) (with respect to \( d\Sigma_{\omega,r} \)) in the support of the measure \( d\Sigma_{\omega,r} \) and almost all \( (\omega,r,p) \in \mathfrak{M} \) (with respect to the induced Lebesgue measure). A complete justification that \( (u_0, v_0) \) satisfies the geometric functional equation in (2.5) if and only if \( (u_0, v_0) \) have the gaussian form in (1.8) requires a multiple-stage argument.

The strategy behind the characterisation is to first argue that \( u_0 \) and \( v_0 \) must be equal (up to non-zero constants), and then establish that \( \hat{u}_0 \) must have a certain amount of regularity. In fact, a delicate geometric argument shows that \( \hat{u}_0 \) must be at least continuous. Once equipped with this information, and furthermore, that \( \hat{u}_0 \) never vanishes, it is possible to solve (2.5) by decomposing \( \hat{u}_0 = fg \) into a product of logarithmically even and odd functions, where

\[
f(\eta) = (\hat{u}_0(\eta)\hat{u}_0(-\eta))^{\frac{1}{2}} \quad \text{and} \quad g(\eta) = \left( \frac{\hat{u}_0(\eta)}{\hat{u}_0(-\eta)} \right)^{\frac{1}{2}}.
\]

The functional equation inherited by \( f \) and \( g \), from \( \hat{u}_0 \), is the classical orthogonal Cauchy functional equation

\[
h(\eta_1 + \eta_2) = h(\eta_1)h(\eta_2)
\]

whenever \( \eta_1 \) and \( \eta_2 \) are orthogonal vectors in \( \mathbb{R}^d \). If \( f \) and \( g \) are normalised so that \( f(0) = g(0) = 1 \), this forces \( f(\eta) = \exp(a|\eta|^2) \) and \( g(\eta) = \exp(b \cdot \eta) \) for some \( a \in \mathbb{C} \) and \( b \in \mathbb{C}^d \), and hence \( \hat{u}_0 \) has the desired form (1.8).

Full details of this argument can be found in [1] as part of a substantial analysis of sharp bilinear estimates of Ozawa–Tsutsumi type.
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