CONTINUITY OF WEIGHTED ESTIMATES FOR SUBLINEAR OPERATORS

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Abstract. In this note we prove that if a sublinear operator $T$ satisfies a certain weighted estimate in the $L^p(w)$ space for all $w \in A_p$, $1 < p < +\infty$, then

$$\lim_{d_w(w_0, w) \to 0} \| T \|_{L^p(w) \to L^p(w)} = \| T \|_{L^p(w_0) \to L^p(w_0)},$$

where $d_w$ is the metric defined in [3] and $w_0$ is a fixed $A_p$ weight.

1. Introduction and notation

We are going to work with positive $L^1_{loc}(\mathbb{R}^n)$ functions $w$ (called weights), that satisfy the following condition for some $1 < p < +\infty$,

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p}} dx \right)^{p-1} < +\infty.$$

The number $[w]_{A_p}$ is called the $A_p$ characteristic of the weight $w$ and we say that $w \in A_p$. The supremum is taken over all cubes $Q$ of $\mathbb{R}^n$.

In [3] the authors defined a metric $d_w$ in the set of $A_p$ weights. For two weights $u, v \in A_p$ we define

$$d_w(u, v) := \| \log u - \log v \|_*,$$

where for a function $f$ in $L^1_{loc}(\mathbb{R}^n)$ we define the $BMO(\mathbb{R}^n)$ norm (modulo constants) as

$$\| f \|_* := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx.$$

The notation $f_Q$ is used to denote the average value of the function $f$ over the cube $Q$ (we will also use the notation $< f >_Q$). In addition, the authors proved that if a linear operator $T$ satisfies the weighted estimate

$$\| T \|_{L^p(w) \to L^p(w)} \leq F([w]_{A_p}),$$

for all $w \in A_p$, where $F$ is a positive increasing function, then for any fixed weight $w_0 \in A_p$ we have

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\[
\lim_{d_*(w,w_0) \to 0} \| T \|_{L^p(w) \to L^p(w)} = \| T \|_{L^p(w_0) \to L^p(w_0)},
\]
which means that the operator norm of \( T \) on the \( L^p(w) \) space is a continuous function of the weight \( w \) with respect to the \( d_* \) metric. In this note we are going to extend this result for sublinear operators \( T \). Namely, we have the

**Theorem 1.** Suppose that for some \( 1 < p < +\infty \), a sublinear operator \( T \) satisfies the inequality

\[
\| T \|_{L^p(w) \to L^p(u)} \leq F([w]_{A_p}),
\]
for all \( w \in A_p \), where \( F \) is a positive increasing function. Fix an \( A_p \) weight \( w_0 \). Then

\[
\lim_{d_*(w,w_0) \to 0} \| T \|_{L^p(w) \to L^p(w)} = \| T \|_{L^p(w_0) \to L^p(w_0)},
\]

Let us mention that the method used in [3] cannot be used for sublinear operators. The argument there does not work for them.

**Remark 2.** In [1] Buckley showed that the Hardy-Littlewood maximal operator defined as

\[
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,
\]
where the supremum is taken over all cubes \( Q \) in \( \mathbb{R}^n \) that contain the point \( x \), satisfies the estimate

\[
\| M \|_{L^p(w) \to L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p-1}},
\]
for \( 1 < p < +\infty \), and all weights \( w \in A_p \), where the constant \( c > 0 \) is independent of the weight \( w \). This means that the assumptions of Theorem 1 hold for \( M \).

We present the proof of the Theorem in the next section.

2. **Proof of Theorem 1**

The main tool for the proof is the inequality (proved in [3])

\[
(1) \quad \| T \|_{L^p(u) \to L^p(u)} \leq \| T \|_{L^p(v) \to L^p(v)} (1 + c[v]_{A_p} d_*(u,v)),
\]
that holds for all \( A_p \) weights \( u, v \in A_p \) that are sufficiently close in the \( d_* \) metric, and for sublinear operators \( T \) that satisfy the assumptions of our Theorem. The positive constant \( c[v]_{A_p} \) that appears in the inequality depends on the dimension \( n \), \( p \), the function \( F \) and the \( A_p \) characteristic of the weight \( v \). Since the quantities \( n, p, F \) are fixed we only write the subscript \( c[v]_{A_p} \) to emphasize this dependence on the characteristic.
Proof. We apply inequality \((1)\) with \(u = w\) and \(v = w_0\) to obtain

\[
\|T\|_{L^p(w) \to L^p(w)} \leq \|T\|_{L^p(w_0) \to L^p(w_0)} (1 + c_{[w_0]} A_p d_* (w, w_0)).
\]

By letting \(d_* (w, w_0)\) go to 0 we get

\[
\limsup_{d_* (w, w_0) \to 0} \|T\|_{L^p(w) \to L^p(w)} \leq \|T\|_{L^p(w_0) \to L^p(w_0)}.
\]

Now it suffices to prove the inequality

\[
\|T\|_{L^p(w_0) \to L^p(w_0)} \leq \liminf_{d_* (w, w_0) \to 0} \|T\|_{L^p(w) \to L^p(w)},
\]

in order to finish the proof. For this reason we use inequality \((1)\) with \(u = w_0\) and \(v = w\)

\[
\|T\|_{L^p(w_0) \to L^p(w_0)} \leq \|T\|_{L^p(w) \to L^p(w)} (1 + c_{[w]} A_p d_* (w, w_0)).
\]

At this point if we know that the constant \(c_{[w]} A_p\) remains bounded as the distance \(d_* (w, w_0)\) goes to 0 we are done.

For this reason we assume that \(d_* (w, w_0) = \delta\) is very close to 0. Then the function \(\frac{w}{w_0}\)
is an \(A_p\) weight with \(A_p\) characteristic very close to 1 (see \([2]\)). How close depends only on \(\delta\), not on \(w\). Thus, if \(R\) is large enough, the weight \((\frac{w}{w_0})^R \in A_p\), with \(A_p\) characteristic independent of \(w\) (again see \([2]\)). Note that from the classical \(A_p\) theory, for sufficiently small \(\epsilon > 0\), we have \(w_0^{1+\epsilon} \in A_p\). Choose the numbers \(R, \epsilon\) such that we have the relation

\[
\frac{1}{R} + \frac{1}{1+\epsilon} = 1, \text{ i.e. such that } R \text{ and } R' = 1 + \epsilon \text{ are conjugate numbers.}
\]

Then, by applying Hölder’s inequality twice we have the following

\[
< w > Q < w^{-\frac{1}{p-1}} >^p_Q = \left\langle \left( \frac{w}{w_0} \right)^{\frac{1}{p-1}} \right\rangle_Q.
\]

\[
\leq \left\langle \left( \frac{w}{w_0} \right) \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^R \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^{-\frac{1}{p-1}} R' \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^{-\frac{1}{p-1}} R' \right\rangle_Q
\]

\[
= \left\langle \left( \frac{w}{w_0} \right) \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^R \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^{-\frac{1}{p-1}} R' \right\rangle_Q \left\langle \left( \frac{w}{w_0} \right)^{-\frac{1}{p-1}} R' \right\rangle_Q
\]

\[
\leq \left[ \left( \frac{w}{w_0} \right) \right]_{A_p} [w_0^{1+\epsilon}]_{A_p} \leq C,
\]

where \(C\) is a constant independent of the weight \(w\). Therefore, \([w]_{A_p} \leq C\).

The last step is to remember how we obtain the constant \(c_{[w]} A_p\) that appears in inequality \((1)\). The authors in \([3]\) used the Riesz-Thorin interpolation theorem with change in measure and then expressed one of the terms that appears in their calculations as a Taylor series. The constant \(c_{[w]} A_p\) appears at exactly this point and it is not difficult to see that it depends continuously on \([w]_{A_p}\). Since this characteristic is bounded for \(w\) close to \(w_0\) in the \(d_*\) metric we have that \(c_{[w]} A_p\) is bounded as well. This completes the proof.

\(\square\)
A consequence of the proof is the following remark.

**Remark 3.** Fix a weight \( w_0 \in A_p \) and a positive number \( \delta \) sufficiently small. There is a positive constant \( C \) that depends on \([w_0]_{A_p}\) and \( \delta \) such that for all weights \( w \) with \( d_*(w, w_0) < \delta \) we have \([w]_{A_p} \leq C\). In addition, from the inequality (see the proof of Theorem 1)

\[
[w]_{A_p} \leq \left( \left( \frac{w}{w_0} \right)^R \right)^{1 \over A_p} [w^{1+\epsilon}]^{1 \over A_p},
\]

and Lebesgue dominated convergence theorem (by letting \( R \to +\infty \) and remembering that the \( A_p \) constant of the weight \( (w/w_0)^R \) is independent of \( R \)) we obtain

\[
\limsup_{d_*(w, w_0) \to 0} [w]_{A_p} \leq [w_0]_{A_p}.
\]

In order to get the remaining inequality

\[
[w_0]_{A_p} \leq \liminf_{d_*(w, w_0) \to 0} [w]_{A_p},
\]

we rewrite (2) as

\[
[w_0]_{A_p} \leq \left( \left( \frac{w_0}{w} \right)^R \right)^{1 \over A_p} [w^{1+\epsilon}]^{1 \over A_p},
\]

and we proceed in the same way as before. In this case the number \( \epsilon \) depends on \([w]_{A_p}\). But we already know that for \( w \) close to \( w_0 \) in the \( d_* \) metric the \( A_p \) characteristic of \( w \) is bounded from above. This means that we are allowed to choose the same number \( \epsilon \) for all weights \( w \) that are sufficiently close to \( w_0 \) and we are done. Therefore, the \( A_p \) characteristic of a weight \( w \in A_p \) is a continuous function of the weight with respect to the \( d_* \) metric.

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**References**


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