Research Statement

Alexander Reznikov

1 Introduction

My research is primarily focused on problems of Harmonic Analysis and their applications to problems of Partial Differential Equations. My dissertation is focused on estimates of different norms of Calderón-Zygmund operators.

  The problems I was solving and I am trying to solve appear to be of a very high interest: different papers and preprints on related topics appear in journals and arXiv one after another. As a young mathematician I try to keep track of different methods for solving these problems, and to compare, combine and improve these methods.

  Since the year 2003 my research was mainly devoted to the following list of topics.

- Sharp constants in the Paneyah-Logvinenko-Sereda theorem, [R1]

- Attainability of infima in different Sobolev Embedding Theorems, [NR1, NR2]

- Sharp estimates for A_p and RH_p weights, [R2, BR1, BR2, RVV]

- Sharp weight estimates for Calderón-Zygmund operators in one weight setting, [RV2, NRV, RTV, NRVV], and in two weight setting, [NRTV1, NRTV2, CURV, NRV1]

- Cauchy independent measures and superadditivity of analytic capacity, [RV]

Below I describe the results in these topics, in which I was involved; and results that I hope to get in the future.

2 Sharp constants in the Paneyah-Logvinenko-Sereda theorem

This was the problem posed by my adviser Professor Viktor Havin in Saint Petersburg State University. The Paneyah-Logvinenko-Sereda theorem says that if a set $S \subset \mathbb{R}$ is sufficiently "thick", then for any function $f \in L^2(\mathbb{R}, dx)$, such that supp $\hat{f} \subset [-\sigma, \sigma]$, the following inequality holds:

$$\int_S |f(x)|^2 dx \geqslant C(S, \sigma) \cdot \int_{\mathbb{R}} |f(x)|^2 dx. \tag{2.1}$$

Here the constant $C(S, \sigma)$ depends on the "thickness" of S and the number σ.

  There is an estimate for a constant $C(S, \sigma)$, however, the estimate is far from being sharp. The leading examples of a "thick" set are

(i) $S = \mathbb{R} \setminus I$, where $I = [-R, R]$ for some $R > 0$,

(ii) $S = \mathbb{R} \setminus \bigcup_{m \in \mathbb{Z}} I_m$, where $I_m = [mk - R, mk + R]$, where $k > 2R > 0$.

For these two types of sets I found the asymptotically sharp dependence of the constant $C(S, \sigma)$ on the parameters R, k, and σ. The answer is the following.

Theorem 2.1. *Let $S = \mathbb{R} \setminus I$, where $I = [-R, R]$ for some $R > 0$. Then the sharp constant $C(S, \sigma)$ in the inequality (2.1) has the following asymptotic:*

$$C(S, \sigma) \sim 1 + \frac{2R\sigma}{\pi}, \quad R\sigma \to 0, \tag{2.2}$$

$$C(S, \sigma) = \frac{e^{2R\sigma}}{4\sqrt{\pi R\sigma}} \left(1 + O(\frac{1}{R\sigma})\right), \quad R\sigma \to \infty. \tag{2.3}$$

Remark 2.2. It is to mention that, due to a rescaling argument, it is enough to consider a constant $C(S, R)$ (the case $\sigma = R$). This constant is equal to $(1 - \|K\|^2)^{-1}$, where $\|K\|$ is the $L^2 \to L^2$ norm of the compact normal operator with discrete spectrum:

$$Ku(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-R}^{R} e^{-ixt} u(t) dt.$$

This operator is strongly connected to a differential operator

$$Tu(x) = \frac{d}{dx}\left((1 - x^2)\frac{d}{dx}u(x)\right), \quad x \in (-1, 1).$$

The theorem is proved by a careful study of eigenfunctions of the operators T and K.

Theorem 2.3. *Let $S = \mathbb{R} \setminus \bigcup_{m \in \mathbb{Z}} I_n$, where $I_m = [mk - R, mk + R]$, where $k > 2R > 0$. Fix an integer n, such that $\frac{n}{2} < \sigma \leqslant \frac{n+1}{2}$ Then the sharp constant $C(S, \sigma)$ in the inequality (2.1) is equal to the fraction $\frac{1}{\lambda_n}$. Here λ_n is the smallest eigenvalue of the matrix*

$$\begin{pmatrix}
c_0 & c_1 & c_2 & \dots & c_{n-1} & c_n \\
\bar{c}_1 & c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
\bar{c}_n & \bar{c}_{n-1} & \bar{c}_{n-2} & \dots & \bar{c}_1 & c_0,
\end{pmatrix}$$

and $c_\ell = \frac{\sin(\ell R)}{R}$.

Remark 2.4. Compare this result to a previous theorem, where it was needed to find an eigenvalue of a compact operator in Hilbert space.

2.1 Open question

Of course, it would be interesting to find the sharp constant $C(S, \sigma)$ for a general "thick" set S.

Definition 2.5. We call a set S "thick" if

$$\gamma = \inf_x \int\limits_S \frac{dt}{1 + (x - t)^2} > 0.$$

Thus, the question is how to get a good estimate of $C(S, \sigma)$ in terms of σ and γ. One can find partial results in [HavJ] and [Pan].

3 Attainability of infima in different types of the Sobolev Embedding Theorem

3.1 The $\mathbb{R}^n$ case

Let us consider a Sobolev Trace Embedding Theorem: for a bounded domain Ω with a strictly Lipschitz boundary and numbers $1 < p < n$ and $p^{\star} = \frac{(n-1)p}{n-p}$, the Trace Embedding Operator $T \colon W_p^1(\Omega) \hookrightarrow L_{p^{\star}}(\partial\Omega)$ is bounded but not compact. Since we lack the compactness, it is a highly non-trivial question whether the infimum

$$\lambda(n, p, \Omega) = \inf_{f \in W_p^1(\Omega)\setminus\{0\}} \frac{\|f\|_{W_p^1(\Omega)}}{\|f\|_{L_{p^{\star}}(\partial\Omega)}}$$

is attained or not.

The following theorem is true.

Theorem 3.1. *Denote*

$$K(n, p) = \inf_{f \in \dot{C}^{\infty}(\mathbb{R}_+^n)\setminus\{0\}} \frac{\|\nabla f\|_{L_p(\mathbb{R}_+^n)}}{\|f(\cdot, 0)\|_{L_{p^{\star}}(\mathbb{R}^{n-1})}},$$

where $\dot{C}^{\infty}(\mathbb{R}_+^n)$ is a set of C^{∞} functions on $\mathbb{R}_+^n$ with bounded support. Then the following holds.

- *For any $\varepsilon > 0$ $K(n, p)$ is attained on a function $w_{\varepsilon}(x) = |x - x_{\varepsilon}|^{-\frac{n-p}{p-1}}$, where $x_{\varepsilon} = (0, 0, \ldots, 0, -\varepsilon)$.*

- *If Ω is a bounded domain with $\partial\Omega \in C^1$, and $\lambda(n, p, \Omega) < K(n, p)$ then the infimum in the definition of $\lambda(n, p, \Omega)$ is attained.*

This theorem tells us that, first of all, to proof the attainability of $\lambda(n, p, \Omega)$ is is enough to present **one** function f, such that

$$\frac{\|f\|_{W_p^1(\Omega)}}{\|f\|_{L_{p^{\star}}(\partial\Omega)}} < K(n, p).$$

Moreover, the theorem hints what function to consider: since $K(n, p)$ itself is attained on a function w_{ε} for **any** $\varepsilon > 0$, we need to adjust the function w_{ε} to our domain. In fact, the following theorem holds.

Theorem 3.2 ([NR1]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a number β, which depends only on Ω, such that for any $1 < p < \frac{n+1}{2} + \beta$ the infimum in the definition of $\lambda(n, p, \Omega)$ is attained.*

This theorem is sharp in terms of smoothness of Ω.

Theorem 3.3. *Let Ω be a polyhedron in $\mathbb{R}^n$. Then there exists a number $\varkappa_0$ such that for any $\varkappa > \varkappa_0$ the infimum in the definition of $\lambda(n, p, \varkappa\Omega)$ is not attained.*

To prove the Theorem 3.2 we fix a point $x_0 \in \partial\Omega$, for which all principar curvatures of $\partial\Omega$ are positive. We use a coordinate system with x_0 being the origin, and $x' = (x_1, \ldots, x_{n-1})$ lying in the tangent plane to $\partial\Omega$ at x_0. The axis x_n is directed into Ω. We introduce a function

$$u_{\varepsilon,}(x) = \varphi(|x'|, x_n) \cdot w_{\varepsilon}(x),$$

where φ is a smooth function with support in a set $\{x \colon |x'| < \rho, 0 < x_n < \rho\}$ and $\varphi(x) = 1$ for any x in $\{x \colon |x'| < \frac{\rho}{2}, 0 < x_n < \frac{\rho}{2}\}$. Moreover, $|\nabla\varphi| < \frac{C}{\rho}$. Then the following lemma holds.

Lemma 3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a number β, which depends only on Ω, such that for any $1 < p < \frac{n+1}{2} + \beta$ and sufficiently small and ε the following inequality holds:*

$$\frac{\|u_{\varepsilon,\rho}\|_{W_p^1(\Omega)}}{\|u_{\varepsilon,\rho}\|_{L_{p^{\star}}(\partial\Omega)}} < K(n, p).$$

3.2 The Riemann Manifold case

In fact, we can define the same $\lambda(n, p, \Omega)$ for an n-dimensional manifold Ω. Then the main theorem reads as follows.

Theorem 3.5 ([NR2]). *Let $n \geqslant 5$ and let Ω be a n-dimensional manifold with C^3-smooth boundary. Suppose that the mean curvature of $\partial\Omega$ with respect to the inner normal is non-positive everywhere. Suppose that there exists a point $y_0 \in \partial\Omega$ such that $\partial\Omega$ is totally geodesic sub-manifold in Ω at y_0, and the scalar curvature of Ω is positive at y_0. Then for some $\beta > 0$, for $2 < p < \frac{n+2}{3} + \beta$, the infimum in the definition of $\lambda(n, p, \Omega)$ is attained.*

It appears that the proof of this theorem is almost the same as the proof of the Theorem 3.2. The difference is that we have to take into account the Rieman metric on Ω and be more careful in calculations.

3.3 Open question

It is still an open question whether the Theorem 3.2 (or its Rieman analog) is sharp in the sense of bounds for p. It is proved in [NR2] that the left bound $p > 2$ in the Theorem 3.5 is sharp. However both right bounds remain unknown. The conjecture is that the answer is positive.

Conjecture 3.6. *For any number $\beta > 0$ there exists a domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^2$, such that the infimum in the definition of $\lambda(n, \frac{n+1}{2} + \beta, \Omega)$ is not attained.*

4 Sharp estimates for properties of A_p and RH_p weights

4.1 Definitions and Notation

The Muckenhoupt weights A_p and Reverse Hölder weights RH_p play a central role in the "one weight" theory of Calderón-Zygmund operators. For any set J we denote

$$\langle w \rangle_J = \frac{1}{|J|} \int_J w(x) dx,$$

where $|J|$ is the Lebesgue measure of J.

Definition 4.1. Let $1 < p < \infty$. A measurable, almost everywhere positive function belongs to a class A_p on an interval I (with endpoints allowed to be infinite) if the following holds:

$$\text{For every interval } J \subset I\colon \ \langle w \rangle_J \left\langle w^{-\frac{1}{p-1}} \right\rangle_J^{p-1} \leqslant Q.$$

The best constant Q is called the A_p *characteristic* of w and is denoted by $[w]_p$.

Definition 4.2. Let $1 < p < \infty$. A measurable, almost everywhere non-negative function belongs to a class RH_p on an interval I (with endpoints allowed to be infinite) if the following holds:

$$\text{For every interval } J \subset I\colon \ \langle w^p \rangle_J^{\frac{1}{p}} \leqslant R \langle w \rangle_J.$$

The best constant R is called the RH_p *characteristic* of w and is denoted by $[w]_{RH_p}$.

As one can see, these definitions do not work for $p = 1$ and $p = \infty$. However, if we carefully consider the limit of the left-hand side, we get the following.

Definition 4.3.   (i) A measurable, almost everywhere positive function belongs to a class A_1 on an interval I (with endpoints allowed to be infinite) if the following holds:

$$\text{For every interval } J \subset I\colon \ \langle w \rangle_J \leqslant Q \inf_J w.$$

The best constant Q is called the A_1 *characteristic* of w and is denoted by $[w]_1$.

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(ii) A measurable, almost everywhere positive function belongs to a class A_∞ on an interval I (with endpoints allowed to be infinite) if the following holds:

$$\text{For every interval } J \subset I\colon \ \langle w \rangle_J \leqslant Q e^{\langle \log(w) \rangle}{}_J.$$

The best constant Q is called the A_∞ *characteristic* of w and is denoted by $[w]_\infty$.

(iii) A measurable, almost everywhere non-negative function belongs to a class RH_1 on an interval I (with endpoints allowed to be infinite) if the following holds:

$$\text{For every interval } J \subset I\colon \ \left\langle \frac{w}{\langle w \rangle_J} \log \frac{w}{\langle w \rangle_J} \right\rangle_J \leqslant R.$$

The best constant R is called the RH_1 *characteristic* of w and is denoted by $[w]_{RH_1}$.

The classes A_p are important as, for example, the Hilbert Transform H is bounded as an operator from $L^p(w)$ to $L^p(w)$ (here $1 < p < \infty$) if and only if $w \in A_p$. We refer the reader to the next section for precise definitions and results.

4.2 Questions about A_p weights

In this section we list the questions concerning A_p weights.

4.3 The class A_∞

It turns out that there are several equivalent definitions of the class A_∞ as a set. These definitions have a corresponding A_∞-characteristics. It is of high importance to relate these characteristics to each other, i.e., get sharp estimates of each of them in terms of others. We start with a well known theorem.

Theorem 4.4. *Consider an interval I and A_p and RH_p classes on this interval. The following statements are true.*

(i) For $p > q$ $A_q \subset A_p$ and $RH_p \subset RH_q$.

(ii) The following chain of equalities holds for sets:

$$A_\infty = \bigcup_{p \geqslant 1} A_p = \bigcap_{p > 1} RH_p = RH_1.$$

(iii) A weight w belongs to A_∞ if and only if for every interval $J \subset I$

$$\langle M(\chi_J \cdot w) \rangle_J \leqslant Q' \langle w \rangle_J.$$

Here M is the Hardy-Littlewood maximal operator. We denote the best Q' by $[w]'_\infty$.

The first part of the theorem is obvious. The second and third parts, however, are not only non-trivial, but pose the following question:

What is the sharp relation between the constants $[w]_\infty$, $[w]'_\infty$, and $[w]_{RH_1}$?

It is known, due to the Stein lemma [St], that $c[w]_{RH_1} \leqslant [w]'_\infty \leqslant C[w]_{RH_1}$. So, it is left to compare the constants $[w]_{RH_1}$ and $[w]_\infty$.

The following theorems were proved by myself and Oleksandra Beznosova, [BR2]. The version of the first theorem was independently proved by T. Hytönen and C. Pérez [HP].

Theorem 4.5. *The following inequality holds for any weight $w \in A_\infty$:*

$$[w]_{RH_1} \leqslant e[w]_\infty.$$

Moreover, the constant e is the best possible.

Theorem 4.6. *If $[w]_{RH_1} = R$ then*

$$[w]_\infty \leqslant Ce^{e^{R+1}-R}.$$

Moreover, this inequality is sharp in R.

The inspiration for proofs of these theorems were works by L. Slavin and V. Vasyunin [Va1, Va2, SlVa]. In fact, one should construct two functions:

$$B_1(x,y) = \sup\left(\langle w\log(w)\rangle_I : \langle w\rangle_I = x,\ \langle\log(w)\rangle_I = y,\ [w]_\infty \leqslant Q\right).$$

and

$$B_2(x,y) = \inf\left(\langle\log(w)\rangle_I : \langle w\rangle_I = x,\ \langle w\log(w)\rangle_I = y,\ [w]_{RH_1} \leqslant R\right).$$

In the paper [BR2] we give an explicit formula for these functions in terms of variables x and y, and the "extremal" examples w, which show sharpness of our estimates.

4.4   The dyadic class A_∞

One can define all classes A_p and RH_p in dyadic setting. In this setting the we consider only intervals J that are dyadic sub-intervals of I. We then get classes A_p^d and RH_p^d. In this case it still holds that $RH_1^d \subset A_\infty^d$, but the opposite is not true. However, the relation between the constants $[w]_{d,\infty}$ and $[w]_{RH_1^d}$. By a very careful modification of the proof of the theorem from [BR2], inspired by the paper of Slavin and Vasyunin [SlVa], we establish the following theorem, proved in [BR1].

Theorem 4.7. *The following inequality holds for any weight $w \in A_\infty^d$:*

$$[w]_{RH_1^d} \leqslant C[w]_{d,\infty}.$$

In the paper [BR1] we give the sharp constant C. It is not e, as it was in the Theorem 4.5.

4.5   Self improvement properties of weights from A_p and RH_p

It was known that the following non-trivial statement holds.

Theorem 4.8. *For any weight $w \in A_p$ there exists a number ε, which depends only on $[w]_p$, such that $w \in A_{p-\varepsilon}$. For any weight $w \in RH_p$ there exists a number ε, which depends only on $[w]_{RH_p}$, sch that $w \in RH_{p+\varepsilon}$.*

In a very motivating paper V. Vasyunin [Va2] was able to establish **sharp** dependence of ε on the A_p and RH_p characteristics of w. However, he did not consider the space RH_1. In our paper with O. Beznosova [BR2] we filled this gap and proved the following theorem.

Theorem 4.9. *Suppose $w \in RH_1$ with $[w]_{RH_1} = R$. Then for every ε, $0 < \varepsilon < \varepsilon_0$ it is true that $w \in RH_{1+\varepsilon}$. The number ε_0 is the smallest solution of the equation*

$$\frac{1}{t} - \log\left(\frac{1}{t}+1\right) = R.$$

Moreover, the dependence of ε_0 on R is sharp.

We essentially follow the Bellman function approach used by V. Vasyunin [Va1, Va2], which is to construct a function

$$B_3(x,y) = \sup\left(\langle w^{1+\varepsilon}\rangle_I : \langle w\rangle_I = x,\ \langle w\log(w)\rangle_I = y,\ [w]_{RH_1} \leqslant R\right),$$

and to find its explicit expression solving a Monge-Ampère equation.

4.6 Distribution functions of the A_p and RH_p weights

It may be useful sometimes to know the worse behavior of a distribution function $\bar{w}(t) = \frac{1}{|I|}|\{x \in I : w(x) \geqslant t\}|$ of a weight w. In the paper [R2] we build the following function B_4 together with extremal examples w. We fix two numbers $-\infty < p_2 < p_1 < \infty$ and consider

$$B_4(x, y, t) = \sup\left(\frac{1}{|I|}|\{x \in I : w(x) \geqslant t\}| : \langle w^{p_1}\rangle_I = x, \ \langle x^{p_2}\rangle_I = y, \ [w]_{p_1, p_2} \leqslant Q\right),$$

where

$$[w]_{p_1, p_2} = \sup_{J \subset I} \langle w^{p_1}\rangle_J^{\frac{1}{p_1}} \langle w^{p_2}\rangle_J^{-\frac{1}{p_2}}.$$

This is a very delicate modification of papers by Slavin and Vasyunin [Va1, Va2, SlVa].

4.7 Open questions

Some questions still remain open. For example, it is not totally clear what happens in the dyadic setting. For example, it is known that $A_\infty^d \not\subset RH_1^d$. However, if we consider only *doubling* weights w, i.e. those with

$$D(w) = \sup \frac{\int\limits_{2J} w}{\int\limits_{J} w} < \infty,$$

then the inclusion holds. We encourage the reader to compare it to the Theorem 4.7, where we **assumed** A_∞^d. Thus, it must be possible to get the sharp dependence of $[w]_{d,\infty}$ on $D(w)$ and $[w]_{RH_1^d}$. There are more similar questions, concerning the dependence of certain quotients on $D(w)$, which remain open, but could be useful, see, for example, the paper by O. Beznosova, J. Moraes and C. Pereyra [BMP].

5 Sharp weight estimates for Calderón-Zygmund operators in one weight setting, [RV2, NRV, RTV, NRVV], and in two weight setting, [NRTV1, NRTV2, CURV]

5.1 Main definitions

In this section we collect main definitions and notation.

Definition 5.1 (Calderón-Zygmund Kernel). A function $K(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a Calderón-Zygmund kernel, if there exist positive numbers C, ε, such that

$$|K(x, y)| \leqslant \frac{C}{|x - y|^n} \tag{5.1}$$

$$|K(x, y) - K(x', y)| \leqslant C\frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}} \text{ if } |x - x'| < \frac{1}{2}|x - y| \tag{5.2}$$

$$|K(x, y) - K(x, y')| \leqslant C\frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}} \text{ if } |y - y'| < \frac{1}{2}|x - y|. \tag{5.3}$$

Definition 5.2 (Calderón-Zygmund Operator). An operator T is called a Calderón-Zygmund operator, if there exists a Calderón-Zygmund Kernel K, such that

$$\text{For any } f \in C_0^\infty, \text{ and any } x \notin \text{supp } f : Tf(x) = \int K(x, y)f(y)dy \tag{5.4}$$

$$T \text{ is a bounded operator from } L^2(dx) \text{ to } L^2(dx). \tag{5.5}$$

The most famous examples of Calderón-Zygmund Kernels are $\frac{1}{x-y}$, that defines a Hilbert Transform, and $\left(\frac{x_1-y_1}{|x-y|^{n+1}}, \ldots, \frac{x_n-y_n}{|x-y|^{n+1}}\right)$, that defines an n-dimensional Riesz Transform.

We now fix a measure μ on $\mathbb{R}^n$ with dimension less than m, $m < n$, i.e., for any ball $B(x,r)$ it holds that $\mu(B(x,r)) \leqslant Cr^m$, where the constant C does not depend on x.

Definition 5.3 (Calderón-Zygmund Kernel of dimension m). A function $K(x,y)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a Calderón-Zygmund kernel of dimension m, if there exist positive numbers C, ε, such that

$$|K(x,y)| \leqslant \frac{C}{|x-y|^m} \tag{5.6}$$

$$|K(x,y) - K(x',y)| \leqslant C\frac{|x-x'|^\varepsilon}{|x-y|^{m+\varepsilon}} \text{ if } |x-x'| < \frac{1}{2}|x-y| \tag{5.7}$$

$$|K(x,y) - K(x,y')| \leqslant C\frac{|y-y'|^\varepsilon}{|x-y|^{m+\varepsilon}} \text{ if } |y-y'| < \frac{1}{2}|x-y|. \tag{5.8}$$

Definition 5.4 (Calderón-Zygmund Operator associated to the measure μ). An operator T_μ is called a Calderón-Zygmund operator associated to the measure μ, if there exists a Calderón-Zygmund Kernel K of dimension m, such that

$$\text{For any } f \in C_0^\infty, \text{ and any } x \notin \text{supp}(fd\mu) : T_\mu f(x) = \int K(x,y)f(y)d\mu(y) \tag{5.9}$$

$$T \text{ is a bounded operator from } L^2(d\mu) \text{ to } L^2(d\mu). \tag{5.10}$$

Finally, if w is a weight, and T is a Calderón-Zygmund Operator (associated to the Lebesgue measure dx), then we denote

$$T_w f(x) = T_{wdx}f(x) = T(fw)(x).$$

Thus, we view T_w as an operator, associated to the measure wdx.

5.1.1 Haar Basis

For simplicity we work with the real line $\mathbb{R}$. For any interval I we define a Haar function h_I by following axioms:

(i) $\text{supp } h_I = I$, and h_I is a constant on the left half and on the right half of the interval I;

(ii) $\int_I h_I(x)dx = 0$;

(iii) $\int_I |h_I(x)|^2 dx = 1$.

We fix some dyadic grid $\mathcal{D}$ on $\mathbb{R}$. It is known that the set $\{h_I\}_{I\in\mathcal{D}}$ is a basis in $L^2(dx)$. Moreover, the set $\{h_I\}_{I\in\mathcal{D}, I\subset I_0} \cup \{\chi_{I_0}\}$ is a basis in $L^2(I_0, dx)$.

Definition 5.5 (Haar Shift). We call an operator $\mathbb{S}$ a (m,n)-Haar shift if

$$\mathbb{S}f(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\subset L, J\subset L \\ |I|=2^{-m}|L|, |J|=2^{-n}|L|}} c_{L,I,J}(f, h_J)h_I(x),$$

where

$$|c_{L,I,J}| \leqslant \frac{\sqrt{|I||J|}}{|L|},$$

and $(f, h_J) = \int f(y)h_J(y)dy$.

Definition 5.6 (Haar Multiplier). We call T_ε a Haar multiplier if T is a $(0,0)$-Haar shift. Here $\varepsilon = \{\varepsilon_I\}$, where $\varepsilon_I = c_{L,I,J}$ in the definition of a Haar shift. Thus,

$$T_\varepsilon f(x) = \sum_{I\in\mathcal{D}} \varepsilon_I(f, h_I)h_I(x), \quad |\varepsilon_I| \leqslant 1.$$

We finish this section with the following theorem, due to Hunt, Muckenhoupt and Wheeden.

Theorem 5.7. *Suppose T is a Calderón-Zygmund Operator. Suppose w is a weight, positive almost everywhere. Let $1 < p < \infty$. Then T_w is bounded as an operator from $L^p(w)$ to $L^p(w^{1-p})$ if $w \in A_{p'}$ — the $A_{p'}$ class, considered in the previous section.*

Remark 5.8. By a change of variables, the operator T_w is bounded from $L^p(w)$ to $L^p(w^{1-p})$ is the same as the operator T is bounded from $L^p(w^{1-p})$ to $L^p(w^{1-p})$. Moreover, it is clear that $w \in A_{p'}$ is equivalent to $w^{1-p} \in A_p$. Thus, one can formulate the Theorem in the following way:

$$w \in A_p \Rightarrow T \text{ is bounded from } L^p(w) \text{ to } L^p(w).$$

5.2 An endpoint result: A_1 conjecture

Besides the last theorem it was known that if the weight $w \in A_1$ then a Calderón-Zygmund operator T is bounded from $L^1(w)$ to $L^{1,\infty}(w)$, an analog of the unweighted end-point result. The A_1 conjecture said that

Conjecture 5.9. *Suppose T is a Calderón-Zygmund operator. Then there exists a constant $C(T)$ such that for any function $f \in L^1(w)$, and for any $t > 0$, the following inequality holds:*

$$|\{x \colon |Tf(x)| \geqslant t\}| \leqslant C(T) \cdot [w]_1 \cdot \frac{\|f\|_{L^1(w)}}{t}.$$

The conjecture says that the norm of T depends on the A_1-characteristic of w linearly.

We would like to emphasize that the A_1 conjecture has very powerful corollaries. For example, if the A_1 conjecture is true, then the A_2 conjecture is also true and follows by an extrapolation method, see [PTV]. For the details on the A_2 conjecture we refer the reader to the Section 5.3.

In the paper [NRVV] we proved the following theorem.

Theorem 5.10. *The A_1 conjecture is false. More precisely:*

(i) *For any sufficiently big number Q there exists a weight $w \in A_1^d$, $[w]_{1,d} = Q$, and a Haar multiplier T_ε, and a function $f \in L^1(w)$, such that*

$$\sup_t t \cdot |\{x \colon |T_\varepsilon f(x)| \geqslant t\}| \geqslant C \cdot Q(\log Q)^{\frac{1}{5}} \cdot \|f\|_{L^1(w)}.$$

(ii) *For any sufficiently big number Q there exists a weight $w \in A_1$, $[w]_1 = Q$, and a function $f \in L^1(w)$, such that the following inequality holds for the Hilbert transform H:*

$$\sup_t t \cdot |\{x \colon |Hf(x)| \geqslant t\}| \geqslant C \cdot Q(\log Q)^{\frac{1}{5}} \cdot \|f\|_{L^1(w)}.$$

Thus, the A_1 conjecture is **false**. Before discussing the proof of this conjecture, we state two related theorems.

Theorem 5.11 (M. Reguera and C. Thiele)**.** *For each number C there is a function w, non-negative almost everywhere, and a function f with compact support, such that*

$$\sup_t t \cdot |\{x \colon |Hf(x)| \geqslant t\}| \geqslant C \int |f(x)| Mw(x) dx.$$

Theorem 5.12 (C. Pérez)**.** *Suppose T is a Calderón-Zygmund operator, and w is a positive almost everywhere weight. Then there exists a constant $C(T)$ such that for any function $f \in L^1(w)$, and for any $t > 0$, the following inequality holds:*

$$|\{x \colon |Tf(x)| \geqslant t\}| \leqslant C(T) \cdot [w]_1 \log(1 + [w]_1) \cdot \frac{\|f\|_{L^1(w)}}{t}.$$

We notice, that, since $w \in A_1 \Rightarrow Mw(x) \leqslant [w]_1 w(x)$, the theorem of Reguera and Thiele follows from our solution to A_1 conjecture. However, we remark that Reguera and Thiele build an example of a non-negative weight w; we, however, guarantee existence of a positive weight, but do not give an example! Ours is the proof of the existence of such an example. In fact, with some efforts the example can be derived from our proof. It is clear now that this will be a highly fractal example.

The theorem of C. Pérez shows that one should look for a sharp endpoint estimate in the logarithmic scale.

5.2.1   Idea of the proof

To work with the A_1 conjecture we consider the following function.

$$B_5(f,t,F,w,m) = \sup\left(|\{x \in [0,1]: T_\varepsilon\varphi(x) \geqslant t\}|: \langle\varphi\rangle_{[0,1]} = f, \langle|\varphi|\omega\rangle_{[0,1]} = F, \langle\omega\rangle_{[0,1]} = w, \inf_{[0,1]}\omega = m, [\omega]_{d,1} \leqslant Q\right).$$

The supremum is taken over all possible functions φ, ω, and Haar Multipliers T_ε. To show the first part of our theorem it is enough to show that, for big values of Q, the following holds:

$$\sup_t(t \cdot B_5(f,t,F,w,m)) \geqslant CQ(\log Q)^{\frac{1}{5}}F.$$

In the paper [NRVV] we are able to do this using certain differential properties of the function B_5. Then, by a method used in [NV], we construct a weight on the real line, that disproves the A_1 conjecture for the Hilbert Transform.

In fact, to prove the needed inequality we used the following properties of the function B_5:

(i) **Homogeneity:** for any number $s > 0$ it is true that

$$sB_5(f,t,\frac{F}{s},\frac{w}{s},\frac{m}{s}) = B_5(f,t,F,w,m), \quad B_5(sf,st,sF,w,m) = B_5(f,t,F,w,m);$$

(ii) **Main inequality:** for sufficiently small numbers df, dF, dw:

$$B_5(f,t,F,w,m) - \frac{1}{4}\Big(B_5(f-df,t-df,F-dF,w-dw,m) + B_5(f+df,t-df,F-dF,w-dw,m) +$$
$$+ B_5(f-df,t+df,F+dF,w+dw,m) + B_5(f+df,t+df,F+dF,w+dw,m)\Big) \geqslant 0;$$

(iii) **Monotonicity:** If $m_+ > m$ then $B_5(f,t,F,w,m) \geqslant B_5(f,t,F,w,m_+)$.

(iv) **More delicate monotonicity:** The function $B_5(f,t,F,w,m) = m\mathbf{B}(\frac{F}{mt},\frac{w}{m},\frac{f}{t})$, and

$$s \mapsto s^{-1}\mathbf{B}(s\alpha, s\beta, \gamma)$$

is increasing for $\frac{|\gamma|}{\alpha} \leqslant s \leqslant \frac{Q}{\beta}$.

(v) **Obstacle condition:** If $|\gamma| < \frac{1}{4}$ then $\mathbf{B}(1,\beta,\gamma) \geqslant \frac{\beta}{8}$.

We notice that from these conditions we can derive that this function B_5 can not have an estimate $B_5(f,t,F,w,m) \leqslant C \cdot Q\frac{F}{t}$. However, it does not give neither the best estimate for B_5, nor the explicit example of weight w and function φ that fail the estimate.

5.2.2   Open questions

Two projects are still ongoing in this area:

(i) **Construct** an example of the weight w and function f that disprove the A_1 conjecture;

(ii) Find a sharp endpoint estimate for the Hilbert Transform in the logarithmic scale, i.e. fill the gap between the lower estimate from [NRVV] and the estimate from above, given by C. Pérez.

5.3 A_2 conjecture in a doubling metric space

The famous A_2 conjecture, solved by T. Hytönen, [H], by now has a lot of different solutions, [HPTV, T, NV1, L, HP, HL, Lac3]. It reads as follows.

Theorem 5.13. *Suppose T is a Calderón-Zygmund Operator. Then there exists a constant $C(T)$, such that for any weight $w \in A_2$ the following estimate holds:*

$$\|T_w\|_{L^2(w) \to L^2(w^{-1})} \leqslant C(T)[w]_2.$$

We notice that the power of the A_2 characteristic here is 1, exactly as in the A_1 conjecture (which, as we explained, is false). We again remark that the truthness of A_1 conjecture would imply the A_2 conjecture. The A_2 conjecture has a long history, see [W, Pet1, Pet2, LPR, CUMP].

We were able to prove the analog of the A_2 conjecture in non-euclidean setting. Suppose (X, ρ) is a metric space with a *doubling condition*: for any $x \in X$ and positive r a ball $B(x, r)$ contains no more than M disjoint balls of radius $\frac{r}{2}$, where M does not depend neither on x, nor on r. It is known, see [KV], that on such space there exists a measure μ with doubling condition: $\mu(B(x, 2r)) \leqslant C\mu(B(x, r))$.

Let $\lambda(x, r)$ be a positive function, increasing and doubling in r, i.e. $\lambda(x, 2r) \leqslant C\lambda(x, r)$, where C does not depend on x and r.

Definition 5.14. $K(x, y) \colon X \times X \to \mathbb{R}$ is a Calderón-Zygmund kernel, associated to a function λ, if there exist positive numbers C, ε, such that

$$|K(x, y)| \leqslant C \min\left(\frac{1}{\lambda(x, \rho(x, y))}, \frac{1}{\lambda(y, \rho(x, y))}\right), \tag{5.11}$$

$$|K(x, y) - K(x', y)| \leqslant C \frac{\rho(x, x')|^{\varepsilon}}{\rho(x, y)^{\varepsilon}\lambda(x, \rho(x, y))}, \quad \rho(x, y) \geqslant C\rho(x, x'), \tag{5.12}$$

$$|K(x, y) - K(x, y')| \leqslant C \frac{\rho(y, y')^{\varepsilon}}{\rho(x, y)^{\varepsilon}\lambda(y, \rho(x, y))}, \quad \rho(x, y) \geqslant C\rho(y, y'). \tag{5.13}$$

Definition 5.15. Let μ be a measure on X, such that $\mu(B(x, r)) \leqslant C\lambda(x, r)$, where C does not depend on x and r. We say that T is a Calderón-Zygmund operator with kernel K if

$$T \text{ is bounded } L^2(\mu) \to L^2(\mu), \tag{5.14}$$

$$Tf(x) = \int K(x, y)f(y)d\mu(y), \ \forall x \notin \operatorname{supp} f d\mu. \tag{5.15}$$

We see that if $X = \mathbb{R}^n$, $\rho(x, y) = |x - y|$, $\lambda(x, r) = r^n$, and $d\mu = dx$, then we get a usual Calderón-Zygmund operator. The A_2 condition now reads as follows.

Definition 5.16. We say that a weight w belongs to A_2 if

$$[w]_2 = \sup_{x, r} \frac{1}{\mu(B(x, r))} \int\limits_{B(x,r)} w d\mu \cdot \frac{1}{\mu(B(x, r))} \int\limits_{B(x,r)} w^{-1} d\mu < \infty.$$

In the paper [NRV] the following theorem is proved.

Theorem 5.17. *Suppose T is a Calderón-Zygmund Operator on a metric space (X, ρ) with doubling condition. Then there exists a constant $C(T)$, such that for any weight $w \in A_2$ the following estimate holds:*

$$\|T_w\|_{L^2(w) \to L^2(w^{-1})} \leqslant C(T)[w]_2.$$

The proof contains following crucial steps.

(i) Constructing an analog of a random dyadic grid, appeared in [NTV] and [H];

(ii) Proving that any Calderón-Zygmund operator T is an "average" of Haar shifts, which is done in [H] and [HPTV] for random dyadic grids in $\mathbb{R}^n$;

(iii) Estimating norms of these (m, n)-Haar shifts with good dependence on parameters (m, n) using a Bellman function approach, [NV1].

We notice that each step is a non-trivial modification of papers by Michael Christ, Tuomas Hytönen, Henri Martikainen, Fedor Nazarov, Sergei Treil, Alexander Volberg. The main obstacle here is that the metric space has balls, but does not have cubes. Thus, it does not have even just one natural "dyadic grid".

5.3.1 Open question

The main question one can ask is the following: what if the measure μ is not doubling? The easiest example is: $X = \mathbb{R}$, but $d\mu = \chi_E dx$ for some set E. Surprisingly, even one of the easiest places of the proof for a doubling measure μ fails for this non doubling example.

5.4 Two weight problem: bump condition

The one weight problem, of its nature, has already two weights. As the reader could notice, in the A_2 conjecture one has a weight $u = w$ and $v = w^{-1}$. The situation is good because these two weights have an obvious property: $uv = 1$. If we lack this property, the situation becomes much more difficult. The question one asks it the following:

What conditions should weights u and v have to assure that the operator T_u is bounded from $L^2(u)$ to $L^2(v)$?

Here T is a Calderón-Zygmund operator.

Thus questions got some attention recently in the works of F. Nazarov, S. Treil, A. Volberg, A. Lerner, M. Lacey, E. Sawyer, C.Y. Shen, I. Uriarte-Tuero, D. Cruz-Uribe, C. Perez, J.M. Martell.

This question was considered for individual operators: Haar shift, see [NTV2, NTV3] and Hilbert Transform, see [NTV4, LSUT, LSSUT, Lac1, Lac2]. The conditions in these questions were formulated in terms of these individual operators.

In our papers [CURV, NRTV1, NRTV2, NRV1] we consider the "bump" approach to this problem. We give a condition on weights u, v which assures that for **any** Calderón-Zygmund Operator T, T_u is bounded from $L^2(u)$ to $L^2(v)$. In [CURV] we also have some L^p results.

The "bump" condition appears from the famous Sarason conjecture.

Conjecture 5.18. *If there exists a number Q, such that for any interval I we have $P_u(z) \cdot P_v(z) \leqslant Q$, then the Hilbert Transform H_u is bounded from $L^2(u)$ to $L^2(v)$. Here*

$$P_u(z) = \int_{\mathbb{R}} \frac{\mathrm{Im}z}{(\mathrm{Re}z - t)^2 + (\mathrm{Im}z)^2} u(t)dt.$$

This conjecture is known to be **false**, see [NV] or [LSUT] for a counterexample.

The bump approach appeared in works of C. Fefferman, [F], Chang-Wilson-Wolf, [CWW]. For more history we refer the reader to the book [CUMP1]. In fact, the A_2 condition for a weight w reads as $\langle w \rangle_I \langle w^{-1} \rangle_I \leqslant Q$. In our setting we have two weights u, v, and the condition $\langle u \rangle_I \langle v \rangle_I \leqslant Q$ is even weaker than the one in the Sarason conjecture (just take $z = c_I + |I|i$, where c_I is the center of the interval I). We notice that $\langle u \rangle_I$ is the squared $L^2(\frac{dx}{|I|})$ norm of the function $u^{\frac{1}{2}}$ on the interval I. We try to consider a stronger norm. Precisely, let $A\colon [0, \infty) \to [0, \infty)$ be a Young function. By $\|u^{\frac{1}{2}}\|_{I,A}$ we denote the normalized Orlitz norm of $u^{\frac{1}{2}}$ on I. In [CURV] we prove the following theorems.

Theorem 5.19. *Suppose $A(t) = t^2 \log^{1+\varepsilon}(t)$. Then if for any interval I*

$$\|u^{\frac{1}{2}}\|_{I,A} \langle v \rangle_I^{\frac{1}{2}} + \|v^{\frac{1}{2}}\|_{I,A} \langle u \rangle_I^{\frac{1}{2}} \leqslant Q,$$

then the operator T_u is bounded from $L^2(u)$ to $L^2(v)$.
The same is true for $A(t) = t^2 \log(t)(\log\log(t))^{1+\varepsilon}$ for sufficiently big ε.

Theorem 5.20. *Suppose $A(t) = t^2 \log^{1+\varepsilon}(t)$. Then if for any interval I*

$$\|v^{\frac{1}{2}}\|_{I,A} \langle u \rangle_I^{\frac{1}{2}} \leqslant Q,$$

then the operator T_u is bounded from $L^2(u)$ to $L^{2,\infty}(v)$.
The same is true for $A(t) = t^2 \log(t)(\log\log(t))^{1+\varepsilon}$ for sufficiently big ε.

In [NRV1] we were able to prove the same result for a wider range of functions A. However, this range does not even include functions $A(t) = t^2 \log(t)(\log\log(t))^{1+\varepsilon}$ for small values of ε. It is still not clear what happens for this function. We suspect that the Theorems **??** will break down for such functions A.

However, in [NRTV1, NRTV2] we showed that the following theorem is true.

Theorem 5.21. *Suppose $\Phi(t)$ is a young function, such that $\frac{1}{\Phi(t)}$ is integrable at ∞. Then if for any interval I*

$$\|u\|_{I,\Phi} \cdot \|v\|_{I,\Phi} \leqslant Q,$$

then the operator T_w is bounded from $L^2(u)$ to $L^2(v)$.

This theorem is the solution of a rather well-known "bump" conjecture. To prove it we used the new *Bellman functional* approach, which was also used in the later paper [NRV1]. We notice that simultaneously this theorem was proved by A. Lerner [L1].

5.4.1   Open questions

There is one open question left: what happens for, say, $A(t) = t^2 \log(t) \log\log(t)(\log\log\log(t))^{100}$? Let us explain why this question is non-trivial. If we denote $\Phi(t) = A(\sqrt{t})$, then the last theorem tells us that

$$\forall I \quad \|u^{\frac{1}{2}}\|_{I,A}\|v^{\frac{1}{2}}\|_{I,A} \leqslant Q$$

implies the boundedness of T_u. However, the question is: does a **weaker**, see [ACUM], condition

$$\forall I \quad \|u^{\frac{1}{2}}\|_{I,A} \langle v \rangle_I^{\frac{1}{2}} + \|v^{\frac{1}{2}}\|_{I,A} \langle u \rangle_I^{\frac{1}{2}} \leqslant Q$$

imply the boundedness of T_u?

We note that in [ACUM] authors build two weights u, v, such that for a function $A(t) = t^2 \log^2(t)$ we have

$$\forall I \quad \|u^{\frac{1}{2}}\|_{I,A} \langle v \rangle_I^{\frac{1}{2}} + \|v^{\frac{1}{2}}\|_{I,A} \langle u \rangle_I^{\frac{1}{2}} < \infty,$$

but

$$\sup_I \|u^{\frac{1}{2}}\|_{I,A}\|v^{\frac{1}{2}}\|_{I,A} = \infty.$$

5.4.2   Three more projects

In this section I would like to explain an alternative approach to the two weight problem.

Universality.   In the paper [RegS] Maria Reguera and James Scurry proved that there exist two weights u, v, such that the Maximal Operator is bounded from $L^2(u)$ to $L^2(v)$, but the Hilbert Transform is not. It is natural to ask a question: is there a natural set $\mathcal{T}$ of operators, such that: if any operator T_u from $\mathcal{T}$ is bounded from $L^2(u)$ to $L^2(v)$, then the Hilbert Transform (and, moreover, all Calderón-Zygmund operators) is also bounded? Two examples of such set $\mathcal{T}$ are the set of all Haar Shifts for all dyadic lattices. Moreover, due to S. Petermichl, [Pet1], for the boundedness of the Hilbert Transform it is enough to consider only (m,n)-shifts for $m + n = 1$.

We ask a question: is it enough to consider the set $\mathcal{T}$ equal to the set of all Haar Multipliers, i.e. all $(0,0)$-shifts? From the result of Petermichl it is clear that we need to prove the following: if all $(0,0)$-shifts are bounded then all $(1,0)$ and $(0,1)$-shifts are bounded.

Remark 5.22. In a one-weight setting the passage from $(0,0)$-shifts to $(0,1)$-shifts and $(1,0)$-shifts is proved in [RTV]. Later S. Treil proved a passage from $(0,0)$-shifts to any (m,n)-shift by a very beatiful Bellman Function argument, see [T].

Two weight testing conditions for a shift in L^p. This questioned was proposed to me by Michael Lacey.

It was proved in [NTV3] that a Haar Shift $\mathbb{S}_u$ is bounded from $L^2(u)$ to $L^2(v)$ if and only if the following holds: there exists a constant C such that for any interval I

$$\|\mathbb{S}_u(\chi_I)\|_{L^2(v)} \leqslant C\|\chi_I\|_{L^2(u)}, \tag{5.16}$$

$$\|\mathbb{S}_v^*(\chi_I)\|_{L^2(u)} \leqslant C\|\chi_I\|_{L^2(v)}. \tag{5.17}$$

However, this question is far from being clear in, say L^4. The best estimate yet is given in the paper [HLM+], and the conditions are not as straightforward. Thus, the question is: can we prove or disprove the sufficiency of test-conditions for the boundedness of $\mathbb{S}_u$ from $L^p(u)$ to $L^p(v)$ for $p \neq 2$?

Two weight testing conditions for the Cauchy transform in L^2. This question was also proposed by Michael Lacey. It is a natural continuation of his papers [Lac1, Lac2], which are the endpoint of the project started in the paper [NTV4].

The Hilbert Transform has a kernel $K(x,y) = \frac{1}{x-y}$ which has an obvious but very important property: $\frac{d}{dy}K(x,y) = \frac{1}{(x-y)^2} > 0$. The Cauchy Operator, however, has a kernel $\frac{1}{z-\xi}$, which lacks this property. Because of that Lacey's proof does not work for this operator. One can ask the same question about Riesz Transform, who's kernel is a vector-values function. Thus, the question we ask is: for classical Calderòn-Zygmund Operators T prove or disprove that

- Joint Poisson A_2 condition for (u,v),
- $\|T_u(\chi_I)\|_{L^2(v)} \leqslant C\|\chi_I\|_{L^2(u)}$, and
- $\|T_v^*(\chi_I)\|_{L^2(u)} \leqslant C\|\chi_I\|_{L^2(v)}$

imply the boundedness of T_u from $L^2(u)$ to $L^2(v)$.

6 Cauchy independent measures

We recall the Definition 5.3: the Cauchy kernel $\frac{1}{\xi-z}$ is a Calderón-Zygmund kernel of dimension $m = 1$. Thus, to define the Cauchy operator $\mathcal{C}_\mu$ properly, we need to work with 1-dimensional measures μ. An example of such measure is the following. Let $E \subset C$ be a compact set. Consider a measure $\mu = H^1|_E$ — a Hausdorff measure on the set E.

Definition 6.1. We call a measure μ a *Cauchy operator measure* if the operator $\mathcal{C}_\mu$ is bounded from $L^2(\mu)$ to itself.

We call a family of measures $\{\mu_j\}$ a *Cauchy independent family* if for a measure $\mu = \sum_j \mu_j$ the following inequality holds with absolute constants C_0, C_1:

$$\|\mathcal{C}_\mu\|_{L^2(\mu) \to L^2(\mu)} \leqslant C_0 \sup_j \|\mathcal{C}_{\mu_j}\|_{L^2(\mu_j) \to L^2(\mu_j)} \leqslant C_1.$$

Such measures μ are important due to their close connection with *analytic capacity*:

$$\gamma(F) = \sup|f'(\infty)|,$$

where the supremum is taken over all analytic functions $f \colon \mathbb{C} \setminus F \to \mathbb{C}$ with $|f| \leqslant 1$ on $\mathbb{C} \setminus F$. Here $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. For a summary of equivalent definitions the reader can see [Tol] and [V]; in these books the reader can also find a lot of applications of Calderón-Zygmund theory. Here F was a compact set in $\mathbb{C}$.

In the paper [RV] we were able to prove the following theorem.

Theorem 6.2. *Let $\mu = \Sigma_j \mu_j$ be as above, and we assume that measures μ_j are supported on compacts E_j lying in the discs D_j such that $20D_j$ are disjoint. We also assume that measures μ_j are extremal in the sense that $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \leq 1$ and $\|\mu_j\| \asymp \gamma(E_j)$. We require that the comparison constants are absolute, which means that they do not depend on j. Let $E = \cup_j E_j$. Then this family is Cauchy independent if and only if for any disc B,*

$$\mu(B) \leq C_0 \gamma(B \cap E). \tag{6.1}$$

Moreover, we showed that the condition $\mu(B) \leq C_0 \gamma(B \cap E)$ is not enough. I.e., the additional structure of μ, posed in the theorem, is needed. To prove this theorem we need the following "super-additivity" property of the analytic capacity, which is of its own interest.

Theorem 6.3. *Let D_j be circles with centers on the real line $\mathbb{R}$, such that for some $\lambda > 1$ it is true that $\lambda D_j \cap \lambda D_k = \emptyset$, $j \neq k$. Let $E_j \subset D_j$ be arbitrary compact sets. Then there exists a constant $c = c(\lambda)$, such that*

$$\gamma(\cup E_j) \geqslant c \sum_j \gamma(E_j).$$

6.0.3 Open questions

The main question that attracts our interest is the following. In all theorems we required certain disjointness of discs D_j. Is it true that the Theorem 6.3 holds for $\lambda = 1$? We were neither able nor to prove it, nor to give a counterexample.

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