Lecture 4 --- Lecture 5

A. Basic Concepts (4.1-4.2)

1. **Experiment:** A process of observing a phenomenon that has variation in its outcome.

   Examples:
   
   (E1). **Rolling a die,**
   (E2). Drawing a card form a shuffled deck,
   (E3). Sampling a number of customers for an opinion survey or
   (E4). Quality inspection of items from a production line.

2. **Outcomes:** The results of an experiment.
   
   Notation: \( e_1, e_2, e_3, \ldots \)

   Examples:  
   E1: looking at the number on the face of the die
   Outcomes: 1,2,3,4,5,6

3. **Sample Space:** The collection of all possible distinct outcomes (elementary outcomes) of an experiment.
   
   Notation: \( S \)

   Examples: E1, \( S=\{1,2,3,4,5,6\} \)

4. **Event:** A subset of the sample space possessing a designated feature.
   
   Notation: \( A, B, C, \) etc.

   Examples: \( A=\{\text{die lands on a face with number less than } 3\}=\{1,2\} \)
   \( B=\{\text{die lands on an odd-numbered face}\}=\{1,3,5\} \)

5. **Occurrence of an event:** An event \( A \) occurs when any one of the outcomes in \( A \) occurs.

   Examples: Rolling a die, we get number 3, then \( B \) happens, \( A \) doesn’t happen.
A little complicated example (Example 1 in page 133): Toss a coin twice and record the outcome head (H) or tail (T) for each toss. Let A denote the event of getting exactly one head, and B the event of getting on heads at all. List the sample space and give the compositions of A and B.

Solution: (Tree diagram)

6. Probability:

Definition 1: The probability of an event is a numerical value between 0 and 1 that can be used to describe the likelihood for this event to happen.

Definition 2: The probability of an event is a numerical value that represents the proportion of times the event is expected to occur when the experiment is repeated under identical conditions.

Notation: The probability of event A is denoted by P(A).

B. Methods of Assigning Probability (4.3)

1. Probability must satisfy

   (1). 0 \leq P(A) \leq 1, for all event A.
   (2). P(A) = \sum_{e \in A} P(e).
   (3). P(S) = \sum_{e \in S} P(e) = 1.
2. Probability assignment in the uniform probability model (in which all elementary outcomes have the same chance to occur)

Example: Tossing a fair coin; Rolling a fair die.

*How to assign probability in the uniform (Equally Likely) model*

- If there are \( k \) elementary outcomes in \( S \), each is assigned the probability \( \frac{1}{k} \).
- \( P(A) = \frac{m}{k} \), Where \( m \) = No. of elementary outcomes in \( A \), \( k \) = No. of elementary outcomes in \( S \).

Example 1: Rolling a die, \( S = \{1,2,3,4,5,6\} \), \( A \) = die lands on a face with number < 3, \( B \) = die lands on an odd numbered face. Find \( P(A) \), \( P(B) \).

Example 2: Tossing a fair coin 2 times. \( S = \{HH, HT, TH, TT\} \). \( A \) = Two heads, \( B \) = At least one head, \( C \) = no head. Find \( P(A) \), \( P(B) \), \( P(C) \).

3. Probability assignment in the non-uniform probability model.

- Relative frequency of \( A \) in \( N \) trials = No. of times \( A \) occurs in \( N \) trials / \( N \)
- We define \( P(A) \) as the value to which the relative frequency stabilizes with increasing number of trials.

C. Event Relations (4.4)

Given an experiment, \( S = \) sample space, \( A \) and \( B \) are two events,

1. Complement of \( A \): \( \bar{A} \) = the set of elementary outcomes that are not in \( A \).
Example:

2. The intersection of A and B: \( A \cap B = AB = \) all outcomes are both in A and B.

Example:

3. The union of A and B: \( A \cup B = \) all outcomes which are either in A or B

Example:
4. The difference of A and B: \( A - B = A \overline{B} \) = all outcomes which are in A, but not in B.

Example:

5. A is a subevent of B, denoted by \( A \subset B \), if all outcomes of A are included in B.

Example:

6. A and B are disjoint, if A and B have no common elementary outcomes. In this case \( A \cap B = \phi \) (empty set).
Example:

Example: Tossing a fair coin 3 times.
(1) Sample Space

S = \{ \}

(2) A = \{HHT, HTH, THH\}, B = \{HHT, HHH, HTH, THH\}

- Find \( \overline{A} \), \( A \cap B \), \( A \cup B \), \( A - B \),

- Is A a subevent of B?

D. Two laws of probability

a) Law of complement.

If \( A \) is an event, then

\[ P(\overline{A}) = 1 - P(A) \]

Example:

b) Addition Law.

If \( A \) and \( B \) are two events, then

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
Example:

Some Special case of Addition Law.

(1). If $A$ and $B$ are disjoint, then $P(A \cup B) =$

Example:

(2). If $A$ is a subevent of $B$, $P(B - A) =$

Example:

Exercise: The probability that a randomly selected student at MSU owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.

i. What is the probability that a student selected at random does not own a bicycle?

ii. What is the probability that a student selected at random owns either a car or a bicycle, or both?
iii. What is the probability that a student selected at random has neither a car nor a bicycle?

E. Conditional Probability

Example 1: You are an analyst, and you want to assess the probability of the event A=“IBM stock price will go up tomorrow”. Consider P(A). The probability you give the event depends on what you know about the company and its performance. If you know more about the company, you may assign a different probability to the event than if you know a little about the company. The probability is conditional upon your information about the company.

Example 2: the probability of A=“you would get 4.0 in STT200”, P(A) will change as some quizzes and exams' grades become available. The updated probability is called “conditional probability”.

Definition (Conditional probability): Conditional probability of an event A given the occurrence of event B is the probability that event A occurs given that event B has already occurred. Notation: P(A|B).

Example 3: Rolling a fair die. Let A = you get an even number, let B= you get 2.
(1) \( P(B) = ? \) \( P(B) = ? \) \( P(AB) = ? \)

(2) If we know for sure, “A has happened”, what is the probability that A happens? That is \( P(A|B) = ? \)

Formula to compute the conditional probability

\[ P(A|B) = \frac{P(AB)}{P(B)} \]
Example (Continuation to Example 3):

Remark: Equivalently, the conditional probability formula can be written as

\[ P(AB) = P(B)P(A|B) \]

This version is called the multiplication law of probability.

Example 4. Suppose \( P(B) = 0.4 \), \( P(\overline{A}) = 0.6 \), \( P(\overline{A} | B) = 0.8 \). Find \( P(B|\overline{A}) \).